VISCOSITY OF THE LIQUID PHASE IN A DISPERSION

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The motion of a dispersion (continuous medium and particles) may be described [1] via the equations of conservation of matter and momentum for the two phases separately. Here it is necessary to know how the viscosity, pressure in the solid, and other quantities vary with the parameters of the motion. This difficulty occurs even for the very simple model where the internal stresses in the dispersed phase are taken as zero, as there is then an uncertainty as to the viscosity of the medium, which is not a material constant and is dependent on the concentration. There is also uncertainty as to the forces of interaction between the phases. There are numerous empirical relationships for these forces, and also a theoretical one [2]. Here an analogous method is applied to derive an expression for the viscosity of the liquid. This viscosity applies to a liquid filtering through a porous medium in the particular case where the concentration is such as to produce close packing of the solid particles. The result corresponds to standard formulas in the case of low concentrations.

We envisaged hindered flow of the medium around the particles on the basis of a cellular model, in which to each particle of radius a there corresponds a spherical cell of radius $b > a$ concentric with the particle, the perturbations caused by the particle being localized in that cell [3]. The surfaces of such cells represent surfaces of symmetry separating the zones of influence of the particles. We equate the volume of a cell to the specific volume of a particle in the system to get

$$b = a\rho^{-1/3}. \tag{1}$$

The general method of [4] will be employed. In the Stokes approximation, the equation for the velocity perturbations in the liquid and particle is

$$\Delta[\nabla \times \mathbf{v}] = 0.$$

Let the flow unperturbed by the particle be defined by

$$v^0 = \alpha_{ik}x_k, \quad \alpha_{ii} = 0, \quad \alpha_{ik} \equiv \alpha_{ki}.$$

As div v = 0, the vectors v may be represented as the rotors of certain axial vectors, and the latter should be linearly dependent on the tensor α_{ik}, so they may be written in a single way, and hence we get for the velocities outside and within a particle

$$\mathbf{v} = \nabla \times [\nabla \times (\boldsymbol{\alpha}\nabla f)], \quad \boldsymbol{\alpha}\nabla f = \alpha_{ik}\,\partial f / \partial x_k,$$

in which f is a scalar function of r. The equation for f,

$$\Delta^2[\nabla \times (\alpha \nabla f)] = 0,$$

has the general solution

$$f = \beta r^6 + \gamma r^4 + \delta r + \varepsilon r^{-1} + \zeta,$$

in which β, γ, δ, ε, and ζ are arbitrary constants.

The velocities in the range $0 \le r \le b$ must be finite; we use a prime to denote quantities referring to the flow within a particle. Then

$$\mathbf{v} = \left(\frac{2}{3}Ar^3 + \frac{B}{r^2} - \frac{5}{3}\frac{C}{r^4}\right)((\boldsymbol{\alpha}\mathbf{n})\mathbf{n})\mathbf{n} + \left(-\frac{5}{3}Ar^3 + \frac{2}{3}\frac{C}{r^4}\right)(\boldsymbol{\alpha}\mathbf{n}),$$

$$\mathbf{v}' = \frac{2}{3}A'r^3((\boldsymbol{\alpha}\mathbf{n})\mathbf{n})\mathbf{n} + \left(-\frac{5}{3}A'r^3 + Dr\right)(\boldsymbol{\alpha}\mathbf{n}),$$

$$p = \mu_0\left(\frac{14}{3}Ar^2 + \frac{2B}{r^3}\right)((\boldsymbol{\alpha}\mathbf{n})\mathbf{n}),$$

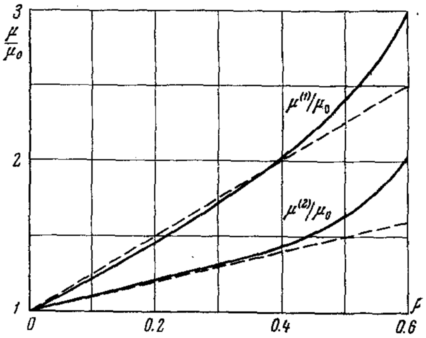
$$p' = \mu'\frac{14}{3}A'r^2((\boldsymbol{\alpha}\mathbf{n})\mathbf{n}) + p_\sigma \tag{2}$$

in which $\mathbf{n} = \mathbf{r}/r$, while p_σ is the change in the pressure discontinuity at the surface of a particle due to the surface tension, the change being due to distortion of this surface $r = a$, where we must have continuity in the normal and tangential components of the total velocities, while the normal component of the velocity must be zero. These quantities are, in general, arbitrary at the surface of a cell.

There have been many discussions concerning the boundary conditions at r = b, and many different conditions have been proposed. Choice of these conditions is one of the major objections to the model [5]. On the other hand, simple physical considerations lead to a single boundary condition at r = b, since the essence of the cell model consists in averaging and smoothing the volumes taken up by the particles, the cell being a measure of this smoothed volume. This averaging is equivalent to the assumption that the particles are identical, as are the cells, which may be justified if the number of particles in the system is sufficiently large; but then symmetry considerations at once show that the radial component of the velocity perturbation must be zero at r = b. Of course, it is clear that the cell model is only an approximation to a real system; but the model gives good results for the forces between the phases and agrees well with empirical relationships [2]. Moreover, a similar model has been used with success in kinetic theory, so we may reasonably expect that this model will give satisfactory agreement with experiment as regards the effective viscosity.

There are thus five arbitrary constants and p_σ (an additional degree of freedom in this problem) to satisfy six boundary conditions, and p_σ appears only in the condition for continuity of the normal stresses at $r = a$; thus the problem can be simplified by discarding this boundary condition, as has been done previously [2, 6], if the derivation of p_σ is of no particular interest. In fact, if we assume that the perturbation of the surface of a particle is small, all the boundary conditions (apart from the condition of continuity for the normal stresses) may be written for an unperturbed spherical surface, since all terms related to surface perturbation are of high orders of smallness. On the other hand, p_σ is of the same order as the other terms in the condition for continuity of the normal stresses, and the condition itself defines the a priori unknown distortion of the surface of a particle. An analogous situation arises in the case of flow around a body coated with a liquid film [6], and also in relation to the forces of phase interaction in a disperse system [2].

So far as we are aware, this feature has never been explicitly pointed out in the literature, which has led to misunderstandings.* We have therefore added an appendix in which we consider more fully the resistance to a drop and the distortion of the drop.



The expressions for the constants in (2) are as follows:

$$Aa^2 = \frac{1}{\Delta(\varkappa, \xi)}\left[(5\varkappa + 2)\xi^5 - 3\varkappa\xi^7\right],$$

$$\frac{B}{a^3} = \frac{1}{\Delta(\varkappa, \xi)}\left[5\varkappa + 2 + 2(\varkappa - 1)\xi^7\right],$$

$$\frac{C}{a^5} = \frac{1}{\Delta(\varkappa, \xi)}\left[3\varkappa + 2(\varkappa - 1)\xi^5\right],$$

$$A'a^2 = D = \frac{1}{\Delta(\varkappa, \xi)}(3 + 7\xi^5 - 5\xi^7),$$

$$\Delta(\varkappa, \xi) = -2(\varkappa + 1) + 7\varkappa\xi^5 - (5\varkappa - 2)\xi^7,$$

$$\xi = a/b = \rho^{1/3}, \qquad \varkappa = \mu'/\mu_0. \tag{3}$$

The motion is completely defined by expressions (3), taken with (2) and with p_σ [found simply from the continuity of the normal stresses, which has not been used in (3)].

The tensor for the mean stresses equals the mean taken over the cell volume for the tensor for the momentum flux density in the system

$$\langle\sigma_{ik}\rangle = -\langle p\rangle\delta_{ik} + \left\langle\mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\right\rangle,$$

where it is readily seen that $\langle p\rangle \equiv 0$, so we have

$$\langle\sigma_{ik}\rangle = \frac{1}{V_0}\left[\mu' \int_{r<a}\left(\frac{\partial v_i'}{\partial x_k} + \frac{\partial v_k'}{\partial x_i}\right)dV + \right.$$

$$\left. + \mu_0 \int_{a<r<b}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)dV\right],$$

in which V_0 is cell volume. Gauss's theorem, with the continuity of the velocity at $r = a$, gives

$$\langle\sigma_{ik}\rangle = \frac{\mu_0}{V_0}\oint_{r=b}(v_i dS_k + v_k dS_i) =$$

$$= 4\pi b^3\alpha_{ik}\frac{\mu_0}{V_0}\left(\frac{2}{3} - \frac{14}{15}Ab^2 + \frac{4}{15}\frac{B}{b^3}\right).$$

We introduce $\xi = a/b$ and note that (1) gives $\xi^2 = \rho$, so

$$\langle\sigma_{ik}\rangle = 2\mu_0\alpha_{ik}\left[1 + \frac{5\varkappa + 2 - {}^{21}/_5\,\varkappa\xi^2 - {}^4/_5(\varkappa - 1)\xi^7}{2(\varkappa + 1) - 7\varkappa\xi^5 + (5\varkappa - 2)\xi^7}\rho\right].$$

Then the effective viscosity $\mu(\rho)$ is

$$\mu(\rho) = \mu_0\left[1 + \frac{5\varkappa + 2 - {}^{21}/_5\,\varkappa\xi^2 - {}^4/_5(\varkappa - 1)\xi^7}{2(\varkappa + 1) - 7\varkappa\xi^5 + (5\varkappa - 2)\xi^7}\rho\right]. \tag{4}$$

In the limit $\varkappa \to \infty$ (suspension of solid particles)

$$\mu^{(1)}(\rho) = \mu_0\left(1 + \frac{5 - {}^{21}/_5\xi^2 - {}^4/_5\xi^7}{2 - 7\xi^5 + 5\xi^7}\rho\right). \tag{5}$$

In the limit $\varkappa \to 0$ (system of gas bubbles in a liquid)

$$\mu^{(2)}(\rho) = \mu_0\left(1 + \frac{2 + {}^4/_5\xi^7}{2(1 - \xi^7)}\rho\right). \tag{6}$$

It is readily seen that (4) in the case $\rho \ll 1$ (dilute systems) becomes Taylor's formula, while (5) and (6) become Einstein's formula and Mark's formula, respectively. The figure shows $\mu^{(1)}/\mu_0$ and $\mu^{(2)}/\mu_0$, where the dashed lines represent the latter two standard formulas.

The $\mu(\rho)$ given by (4)–(6) is the effective viscosity for a liquid passing through a porous body with porosity $\varepsilon = 1 - \rho$. The total momentum transfer in the system consists of several parts, which are, in general, interdependent:

1) momentum transfer within the liquid;

2) momentum transfer related to local pulsation movements of the particles;

3) momentum transfer due to friction between particles, inhomogeneity in the external forces, etc.

It is clear that relations (4)–(6) describe only the part of the momentum transfer associated with movement of the liquid phase; $\mu(\rho)$ coincides with the viscosity of the system as a whole only when the latter two components in the momentum transfer are negligible relative to the first. Certain results [9] allow one to elucidate the necessary conditions for smallness of the second component. The third component may be considered as small if the concentration is not too close to the close-packing concentration.

It appears to be usual in most of the papers known to us to ignore the difference between the viscosity of the liquid phase and the total effective viscosity of the system; in particular, this is so in most papers on the cell model. For this reason, some results, in particular Simha's formula [3], relate to neither viscosity.

Most experiments [7] on suspensions of moderate concentration fail to meet the conditions for agreement between the viscosities of the liquid phase and the suspension, so these experiments are not suitable for testing (4)–(6); but there are data (e.g., [8] on the creep of solutions of concrete) that indicate that Einstein's formula applies approximately not only for ρ from 0 to 2–3% (as is usually assumed) but also for

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*See, for example, Slezkin's abstract of [6] (Ref. Zh. Mekhanika, no. 6, B648, 1966).

all ρ up to 50–60%. The densities of the phases were approximately identical under the conditions of these experiments, and Archimedes' number was extremely small, mainly on account of the high viscosity of the cement base that served as the dispersion medium. The conditions of smallness for the internal stresses in the dispersed phase were therefore met for nearly all ρ, and it is possible to compare the experiments with the $\mu(\rho)$ of (4)–(6). The figure shows that $\mu(\rho)$ can actually be found from Einstein's formula over a wide range in ρ.

There is also the anomalous-viscosity effect, in which the measured viscosity of a suspension is somewhat less than the value indicated by Einstein's formula. This corresponds to the region of ρ not too large in the figure.

The $\mu(\rho)$ of (5) characterizes the viscous behavior of a filtering liquid and defines the viscous term that can be added when necessary to the Darcy filtration equations. In particular, a term of the form

$$\mu(\rho)\,\partial^2 u_i / \partial x_j \partial x_j$$

may be entered in the equations of relative motion (interphase slip) for a fluidized bed [10].

APPENDIX

Distortion of a moving drop in a viscous liquid. We solve the equations of motion within and outside the drop, neglecting inertial terms, to get

$$v_r = \left(\frac{b_1}{r^3} + \frac{b_2}{r} + u\right)\cos\theta,$$

$$v_\theta = \left(\frac{b_1}{2r^3} - \frac{b_2}{2r} - u\right)\sin\theta, \qquad p = \mu b_2 / r^2 \cos\theta,$$

$$v_r' = (a_1 r^2 + a_2)\cos\theta, \qquad v_\theta' = (-2a_1 r^2 - a_2)\sin\theta,$$

$$p' = \mu' 10 a_1 r \cos\theta$$

in which primes denote quantities for flow within the drop and u is the speed of the liquid at an infinite distance from the drop. The drop may be considered as spherical to small quantities of the first order, and we get the following results from the conditions for continuity of the velocity, the tangential component of the stress, and zero radial motion at the surface of the drop:

$$\frac{b_1}{R^3} = \frac{u}{2}\frac{\mu'}{\mu+\mu'}, \qquad \frac{b_2}{R} = -\frac{u}{2}\frac{3\mu'+2\mu}{\mu+\mu'},$$

$$a_1 R^2 = \frac{u}{2}\frac{\mu}{\mu+\mu'}, \qquad a_2 = -\frac{u}{2}\frac{\mu}{\mu+\mu'}.$$

Simple steps then give the standard formula for the resistance; but the condition for continuity in the normal stresses at $r = a$ is not obeyed, and at that point we have the additional pressure discontinuity

$$p_\sigma = p - p' = \frac{9}{2}\frac{u}{R}\frac{\mu\mu'}{\mu+\mu'}\cos\theta.$$

We assume that the equation for the perturbed sphere is $r = a + \zeta(\theta)$, $\zeta \ll a$, to express p_σ via derivatives with respect to ζ, whereupon

$$\frac{d^2\zeta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\zeta}{d\theta} + 2\zeta = -\frac{9}{2}\frac{uR}{\sigma}\frac{\mu\mu'}{\mu+\mu'}\cos\theta.$$

The general solution of this equation is

$$\zeta(\theta) = \left[\frac{3}{2}\frac{uR}{\sigma}\frac{\mu\mu'}{\mu+\mu'}\ln\sin\theta + \frac{s_1}{2}\ln\frac{1-\cos\theta}{1+\cos\theta} + \frac{s_1}{\cos\theta} + s_2\right]\cos\theta \begin{pmatrix} s_1 = \text{const} \\ s_2 = \text{const} \end{pmatrix}.$$

We choose s_1 so that $\zeta(\theta)$ has no singularities at θ equal to 0 and π to get

$$\zeta(\theta) = \frac{3}{2}\frac{uR}{\sigma}\frac{\mu\mu'}{\mu+\mu'} \times \cos\theta\left[\ln\sin\theta - \frac{1}{2}\ln\frac{1-\cos\theta}{1+\cos\theta} + \frac{1}{\cos\theta}\right] + s_2\cos\theta,$$

while s_2 is determined from the condition of constant volume of the drop:

$$\int_0^\pi (a + \zeta(\theta))^3 \sin\theta\, d\theta = 2a^3.$$

The condition for continuity of the normal stress thus defines the distortion of the drop and has no application to the derivation of v, v', p, or the formula for the resistance. This feature may be formally taken into account by putting $p' = \mu'(10a_1 r + a_3)\cos\theta$, i.e., by introducing a new constant a_3 into the expression for p', as has been done previously [2, 6].

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